ON NONUNIQUENESS OF THE STATIONARY SOLUTION OF THE SYSTEM OF EQUATIONS OF THE THEORY OF COMBUSTION

(O NEEDINSTVENNOSTI STATSIONARNOGO RESHENIIA SISTEMY URAVNENII TEORII GORENIIA)

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As is known, the one-dimensional combustion process of a gas mixture is described by a nonlinear system of partial differential equations of the form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[\alpha (U) \frac{\partial U}{\partial x}\right] + F(U) C, \qquad \frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[\alpha_1 (U) \frac{\partial C}{\partial x}\right] - F(U)C \qquad (0.1)$$

$$F(U) \equiv 0, \quad u \in [0, U_0], \quad F(U) > 0, \quad U > U_0$$

Here U is the mixture temperature, $C \geq 0$ the concentration of active substance, $F(U)C$ the reaction rate, $\alpha(U) > 0$ the coefficient of heat conduction, $\alpha_1(U) > 0$ the coefficient of diffusion.

Let us seek the solution of a special kind of system, called stationary

$$U = u(y), \quad C = c(y), \quad y = x + \lambda t, \quad \lambda = \text{const} > 0$$

which satisfies the conditions

$$u(-\infty) < u(y) < u(\infty), \qquad c(-\infty) > c(y) > c(\infty)$$

The system (0.1) hence becomes

$$\lambda \frac{du}{dy} = \frac{d}{dy}\left[\alpha(u) \frac{du}{dy}\right] + F(u) c, \qquad \lambda \frac{dc}{dy} = \frac{d}{dy}\left[\alpha_1(u) \frac{dc}{dy}\right] - F(u) c \qquad (0.2)$$

It is easy to prove that $u'(y) > 0$ for all y. Let us prescribe the following conditions for the solution of the system (0.2):

$$u(-\infty) = 0, \qquad c(-\infty) = c_0 > 0, \qquad c(\infty) = 0$$

It follows from the existence of $u(\pm\infty)$ and $c(\pm\infty)$ that $c'(\pm\infty) = u'(\pm\infty) = 0$ if the latter exist. We have from (0.2)

$$\lambda [c(y) + u(y) - c_0 - u(-\infty)] = \alpha(u)\, du / dy + \alpha_1(u)\, dc / dy$$

Hence, in turn

$$c(-\infty) + u(-\infty) = c(\infty) + u(\infty), \quad \text{for} \quad u(\infty) = u_+ = c_0$$

Taking into account that $u' > 0$, we take u as independent variable. Let us introduce the notation

$$v(u) = \alpha(u)\, du / dy > 0, \quad \alpha(u) F(u) = f(u)$$

$$f(u) > 0, \quad u > u_0, \quad f(u) \equiv 0, \quad u \in [0, u_0], \ \alpha(u) / \alpha_1(u) = \beta(u) > 0 \tag{0.3}$$

Consequently, the system (0.2) becomes

$$v = \lambda - \frac{f(u)c}{v}, \quad c = \beta(u)\left[\frac{\lambda}{v}(c + u - u) - 1\right] \tag{0.4}$$

with the conditions $v(0) = 0$, $v(u_+) = \sigma(u_+) = 0$.

Since $f(u) \equiv 0$, $u \in [0, u_0]$, the latter is then equivalent to

$$v(u_0) = \lambda u_0 \tag{0.5}$$

It is required to determine $\sigma(u)$ and $v(u)$ in $[0, u_+]$ (thereby $u(y)$ and $\sigma(y)$ will be determined to the accuracy of a parallel transfer along the y-axis as well as the constant λ

The existence of a solution of the system (0.4), (0.5) has been proved in [2] in the particular case of $\beta(u)$ = const. It has also been proved that the solution of this system is unique for $\beta(u)$ = const > 1. The question of uniqueness in the general case therefore remains open. Another particular case with $\beta \equiv 1$ reduces the system (0.4), (0.5) to the single equation considered in [1] (where the existence and uniqueness of the solution was proved), and also in [3 and 4].

In this connection, the assumption existed that the system (0.4), (0.5) has a unique solution for any $f(u)$ and $\beta(u)$ satisfying the constraints (0.3) By constructing a contradictory example, it is proved herein that uniqueness even may not hold despite compliance with (0.3).

Let us assume that for some combination of values u_0, u_+ and the functions $f(u)$, $\beta(u)$ the system (0.4), (0.5) has two solutions $v_i(u)$, $\sigma_i(u)$ (i = 1,2). Let us introduce the notation

$$a(u) = c_2(u) / c_1(u), \tag{0.6}$$

$$b(u) = v_2(u) / v_1(u) \tag{0.7}$$

for $u \in (u_0, u_+)$. The values of $a(u_+)$ and $b(u_+)$ are determined by a passage to the limit.

Let us find $f(u)$, $\beta(u)$ and u_+ in terms of u_0, λ_1, λ_2, $a(u)$, $b(u)$. To do this let us first form a system of differential equations to determine $v_1(u)$, u_+ and $\sigma_1(u)$ in terms of u_0, λ_1, λ_2, $a(u)$, $b(u)$. To do this let us first form a system of differential equations to determine $v_1(u)$, u_+ and $\sigma_1(u)$ in terms of u_0, λ_1, λ_2, $a(u)$ and $b(u)$. After transformation we have from (0.6) and (0.7)

$$c_1' = \frac{a'c_1[\lambda_1(c_1 + u - u_+) - v_1]\, b}{(\lambda_2 - \lambda_1 b)\, ac_1 + (\lambda_2 - \lambda_1 ab)(u - u_+) + (a - 1)\, bv_1} \tag{0.8}$$

As will be proved below, (0.7) has singular points, which precludes assignment of the initial condition. From (0.6) and (0.5) follows

$$b(u) = \lambda_2 / \lambda_1, \ u \in [0, u_0].$$

Evidently the function $b(u)$ is continuously differential in $(0, u_+)$. Therefore, $b'(u_0) = 0$. We have $b(u) > 0$ in $[0, u_+)$. Substituting both the assumed solutions into (0.4), eliminating $f(u)$ and utilizing (0.6), we obtain

$$\frac{(\lambda_2 - v_2')\, v_2}{(\lambda_1 - v_1')\, v_1} = \frac{c_2}{c_1} = a(u), \qquad u \in (u_0, u_+) \tag{0.9}$$

Hence, according to (0.6) we obtain an equation to determine $v_1(u)$

$$v_1' = \frac{bb'}{a - b^2} v_1 + \frac{a\lambda_1 - b\lambda_2}{a - b^2}, \quad v_1(u_0) = \lambda_1 u_0 \quad \text{for} \quad v_1' - \lambda_1 = \frac{b(b'v_1 - \lambda_2 + b\lambda_1)}{a - b^2} \tag{0.10}$$

Evidently (0.8) may be solved independently of (0.7). From (0.4), (0.3) and also the constraints imposed on $\sigma(u)$ and $v(u)$ it follows that $v_1' - \lambda_1 < 0$ that is

$$\frac{b'v_1 - \lambda_2 + b\lambda_1}{a - b^2} < 0 \qquad (0.11)$$

Now let $\lambda_1, \lambda_2, a(u), b(u), u \in [u_0, \infty)$ be assigned in advance, but not obtained as a result of solving (0.4), (0.5). Moreover, as before, u_0 is prescribed. Solving (0.7) (0.8), we can obtain $v_1(u)$, u_+, $\sigma_1(u)$ and $f(u)$ and $\beta(u)$ in terms of them. The following conditions should hence be satisfied:

1) $v_1(u)$ should vanish at least for $u > u_0$. The point of intersection with the horizontal axis nearest to u_0 will be taken as u_+, after which the segment (u_+, ∞) is excluded from the considerations.

2) At least one continuous solution of (0.8), which vanishes at $u = u_+$, should exist in $[u_0, u_+]$.

3) The $f(u)$ and $\beta(u)$ obtained should be continuous, differentiable, and satisfy the conditions $f(u_0) = 0$, $f(u) > 0$ for $u \in (u_0, u_+]$.

As regards the semi-interval $[0, u_0)$, $f(u) \equiv 0$ has already been determined therein; any positive function differentiable in $[0, u_0)$ as well as the juncture point $u = u_0$ may be taken as $\beta(u)$.

The functions $a(u)$ and $b(u)$ are constructed in Section 1; $v_1(u)$ is determined in Section 2 and the existence of u_+ is proved; in Section 3 it is proved that $v_1' - \lambda_1 < 0$ in $(u_0, u_+]$, which is necessary to the proof for $f(u)$ being positive in this semi-interval; $\sigma_1(u)$ is determined in Section 4. The equation (0.7) has two singular points, one of which is $(u_+, 0)$, in the $\sigma_1 u$ plane. The existence of a single integral line passing through both singular points is proved. It is proved that $\sigma_1'(u) < 0$ points of the mentioned line. This is used to prove that $\beta(u)$ is positive.

1. Let us establish the sufficient conditions which should be imposed on $f(u)$ and $\beta(u)$ in order that the listed requirements be satisfied. Let us take an arbitrary $u_1 > u_0$ and let us construct any twice continuously differentiable function $b(u)$ in $[u_0, u_1]$ which will satisfy the following conditions:

$$b(u_0) = \lambda_2 / \lambda_1, \quad b'(u_0 + 0) = 0, \quad b'(u) < 0, \quad u \in (u_0, u_1) \qquad (1.1)$$
$$b'(u_1 - 0) = 0, \qquad b(u_1) \in (0, \lambda_1 / \lambda_2)$$

Further, let us select an arbitrary $B \in (\lambda_1 / \lambda_2, 1)$ and let us construct any twice continuously differentiable function $b(u)$ in (u_1, ∞), which will satisfy the conditions

$$b(u_1 + 0) = b(u_1), \quad b'(u_1 + 0) = 0, \quad b''(u_1 + 0) = b''(u_1 - 0), \quad \lim_{u \to \infty} b(u) = B$$
$$0 < b'(u) < \frac{\lambda_2 - \lambda_1}{\lambda_1 u}, \qquad u \in (u_1, \infty) \qquad (1.2)$$

Evidently, a function $b(u)$ satisfying the last inequality and at $u \to \infty$ tending to any value greater than $b(u_1)$, (in particular to the selected B), may be chosen. The possibility of satisfying the remaining conditions is evident.

Let us also construct some twice continuously differentiable function $a(u)$ in $[u_0, u_1]$, which will satisfy the following conditions

$$a(u_0) \in \left(\frac{\lambda_2^2}{\lambda_1^2}, \frac{\lambda_2}{\lambda_1 b(u_1)}\right), \quad a(u_1) \in \left(a(u_0), \frac{\lambda_2}{\lambda_1 b(u_1)}\right), \quad a'(u) > 0 \qquad (1.3)$$
$$u \in [u_0, u_1), \quad a'(u_1 - 0) = 0, \quad a''(u_1 - 0) < 0$$

Let us chose an arbitrary

$$A \in (\max\{1, a(u_1) b(u_1) B\}, B\lambda_2 / \lambda_1)$$

Let us construct some twice continuously differentiable function $a(u)$ in (u_1, ∞), which will satisfy the following conditions

$$a(u_1 + 0) = a(u_1), \qquad a'(u_1 + 0) = 0, \qquad a''(u_1 + 0) = a''(u_1 - 0)$$
$$a'(u) < 0, \quad u \in (u_1, \infty), \quad \lim_{u \to \infty} a(u) = A, \qquad a(u) < \frac{A}{Bb(u)} \qquad u \in [u_1, \infty) \qquad (1.4)$$

The existence of functions satisfying the first four conditions of (1.4) is obvious. Moreover, it follows from the above that:

$$A < A / B < \lambda_2 / \lambda_1 < \lambda_2^2 / \lambda_1^2 < a(u_1)$$

Hence, the existence of functions satisfying the fifth condition of (1.4) results. The possibility of also satisfying the last condition of (1.4) follows from the fact that the function $A/Bb(u)$ decreases monotonously as u changes from u_1 to ∞ respectively

from $\frac{A}{Bb(u_1)} > a(u_1)$ to $\frac{A}{Bb(\infty)} = \frac{A}{B^2} > A = a(\infty)$

It follows from the construction of the functions $a(u)$ and $b(u)$ that both are twice continuously differentiable in $[u_0, \infty)$, and particularly at the point u_1, where the matching has been made.

Let us prove the existence of an $h > 0$ such that the inequality

$$a(u) - b^2(u) \geqslant h \tag{1.5}$$

is valid for the constructed functions $a(u)$ and $b(u)$ in $[u_0, \infty)$

Let $u \in [u_0, u_1]$. Then it follows from (1.1) and (1.3) that:

$$a(u) - b^2(u) \geqslant a(u_0) - b^2(u_0) = a(u_0) - \left(\frac{\lambda_2}{\lambda_1}\right)^2 > 0$$

Now, let $u \in (u_1, \infty)$. Then it follows by virtue of (1.2) and (1.4) that $a(u) - b^2(u) > A - B^2 > 0$. Putting $h = \min\{a(u_0) - (\lambda_2/\lambda_1)^2, A - B^2\}$, we obtain (1.5).

Moreover, let us prove the existence of an $H > 0$ such that for all $u \in [u_0, \infty)$

$$a(u) - b^2(u) \leqslant H \tag{1.6}$$

In fact, the function $a(u)$ takes on its maximum value at $u = u_1$, and $b(u)$ its minimum. Therefore, $a(u) - b^2(u) \leq a(u_1) - b^2(u_1)$.

Putting $H = a(u_1) - b^2(u_1)$, we obtain (1.6). Let us also note that

$$b(u) < \lambda_2 / \lambda_1, \quad u > u_0, \quad a(u) > 1, \quad u \geqslant u_0, \quad b(u) < 1, \quad u \geqslant u_1 \tag{1.7}$$

2. By constructing $a(u)$ and $b(u)$, $u \in [u_0, \infty)$, in this manner, we determine $v_1(u)$ from (0.9) under the initial condition $v_1(u_0) = \lambda_1 u_0$

$$v_1(u) = X(u)\, Y(y) \qquad \left(X(u) = \exp \int_{u_0}^{u} \frac{bb'}{a - b^2}\, ds, Y(u) = \int_{u_0}^{u} \frac{\lambda_1 a - \lambda_2 b}{(a - b^2)\, X(s)}\, ds + \lambda_1 u_0\right) \tag{2.1}$$

Let us prove the existence of a $p > 0$ such that for all $u \in [u_0, \infty)$ we have

$$X(u) \leqslant p \tag{2.2}$$

Let $u \in [u_0, u_1]$. By virtue of (1.1) and (1.5) we have $X'(u) \leq 0$. Hence, $X(u) \leq X(u_0)$. Now, let $u \in (u_1, \infty)$ Then

$$X(u) = X(u_1) \exp \int_{u_1}^{u} \frac{bb'}{a - b^2}\, ds \leqslant X(u_1) \exp \frac{1}{2h}[b^2(u) - b^2(u_1)] <$$

$$< X(u_1) \exp \frac{1}{2h}[B^2 - b^2(u_1)]$$

Putting

$$p = \max\left\{X(u_0),\ X(u_1) \exp \frac{1}{2h}[B^2 - b^2(u_0)]\right\}$$

we obtain (2.2). Taking into account that $X'(u) > 0$ for $u > u_1$, we obtain the existence of $X(\infty)$. Let us note that since we have $\lambda_1 a - \lambda_2 b \geq \lambda_1 a(u_0) - \lambda_2 b(u_0) > 0$ for $u \in [u_0, u_1]$, then $v_1(u) > 0$ on this segment.

Let us now prove the existence of a $u_+ > u_1$ such that $v_+ = 0$, $v(u) > 0$, $u < u_+$

Let us consider $Y'(u)$ in $[u_1, \infty)$. Evidently $Y'(u_1) > 0$. As u increases between u_1 and ∞ the function $\lambda_1 a - \lambda_2 b$ will decrease monotonously by virtue of the above, and

$$\lim_{u\to\infty} (\lambda_1 a - \lambda_2 b) = \lambda_1 A - \lambda_2 B < 0$$

Hence, $Y'(u)$ changes sign at some point $u = u_2 > u_1$, and the function $Y(u)$ will decrease monotonously for $u > u_2$. It can not emerge beyond the horizontal asymptote since

$$\lim_{u\to\infty} Y'(u) = \frac{\lambda_1 A - \lambda_2 B}{(A - B^2)\, X(\infty)} < 0$$

Hence, the existence of the desired point u_+ has been established.

Let us note that $\lambda_1 a(u_+) - \lambda_2 b(u_+) < 0$ has been proved in passing. We therefore have $v'(u_+) < 0$ from (0.8). We shall carry out all the subsequent disscussion for just $u \le u_+$.

3. Let us prove that

$$v_1' - \lambda_1 < 0, \qquad u \in (u_0, u_+) \tag{3.1}$$

Taking account of (1.1), (1.5) and (1.7), compliance with (3.1) in $(u_0, u_1]$ follows from (0.10). By virtue of (0.5) we have

$$v_1(u) < \lambda_1 u, \qquad u \in (u_0, u_1]$$

Now, let us prove that $v_1(u) < \lambda_1 u$ also in $(u_1, u_+]$. Let us assume the opposite, i.e. a $u = u_2 \in (u_1, u_+]$ is found such that $v_1(u_2) = \lambda_1 u_2$.

If the mentioned point is not unique, then u_2 is taken to be the closest to u_1 . Then, according to the Lagrange theorem, a $u_3 \in (u_1, u_2)$ is found such that

$$v_1'(u_3) = \frac{\lambda_1 u_2 - v_1(u_1)}{u_2 - u_1} > \lambda_1$$

On the other hand, since $v_1(u_3) < \lambda_1 u_3$, then taking account of (1.2) and (0.9) we will have

$$v_1'(u_3) - \lambda_1 = \frac{b(b'v_1 - \lambda_2 + b\lambda_1)}{a - b^2} < \frac{(v_1/\lambda_1 u - 1)(\lambda_2 - \lambda_1 b)\, b}{a - b^2} < 0$$

which is impossible. Now, utilizing (1.2) we obtain

$$b'v_1 - \lambda_2 + b\lambda_1 < b'\lambda_1 u - \lambda_2 + b\lambda_1 < 0$$

Hence, according to (0.9) we have

$$v_1' - \lambda_1 < 0, \qquad u \in (u_1, u_+)$$

Therefore, (3.1) has been proved.

4. Having determined the function $v_1(u)$ on $[u_0, u_+]$ in such a manner, and $v_2(u)$ thereby (since $b(u)$ is known), let us give the determination of $c_1(u)$. Let us consider (0.7) in the domain E (see Fig.1)

$$u \in [u_0, u_+],\ c_1 \in [0, L(u)],\ L(u) = v_1/\lambda_1 + u_+ - u$$

By virtue of the proved properties of the Function $v_1(u)$, we have

$$L'(u_0) = 0, \qquad L'(u) < 0, \qquad u \in (u_0, u_+] \tag{4.1}$$

It is also evident that

$$L(u) > 0, \qquad u \in [u_0, u_+),\ L(u_+) = 0$$

Let us rewrite (0.8) as

$$c_1' = \frac{a'\varphi(u, c_1)}{\psi(u, c_1)}, \qquad \varphi(u, c_1) = c_1[\lambda_1(c_1 + u - u_+) - v_1]\, v_2 \tag{4.2}$$

$$\psi(u, c_1) = (\lambda_2 v_1 - \lambda_1 v_2)\, a c_1 + (\lambda_2 v_1 - a\lambda_1 v_2)(u - u_+) + (a - 1)\, v_1 v_2$$

Evidently we have $\varphi(u, \sigma_1) = 0$ on the upper $\sigma_1 = L(u)$ and lower $\sigma_1 = 0$ boundaries of the domain E. At inner points and in the interval $u = u_0$, $0 < \sigma_1 < L(u_0)$ we have $\varphi(u, \sigma_1) < 0$. Let us consider the behavior of ψ in the domain E. After transformations, we have on the upper boundary according to (1.7) and (1.1)

$$\psi\,[u, L(u)] = (u_+ - u)\,\lambda_2 v_1 (a - 1) + v_1^2\,(a\lambda_2 / \lambda_1 - b) > 0, \qquad u \in [u_0, u_+)$$

After transformations we have on the lower boundary

$$\psi(u, 0) = [\lambda_1 (u_+ - u) + v_1] \left[a(u)\, b(u) - \frac{\lambda_2 (u - u_+) - v_2}{\lambda_1 (u - u_+) - v_1} \right] v_1$$

According to the Cauchy theorem, a $u^* \in (u, u_+)$ may be found such that

$$\frac{\lambda_2 (u - u_+) - v_2(u)}{\lambda_1 (u - u_+) - v_1(u)} = \frac{\lambda_2 - v_2'(u^*)}{\lambda_1 - v_1'(u^*)} = \frac{a(u^*)}{b(u^*)}$$

Hence

$$\psi(u, 0) = [\lambda_1 (u_+ - u) + v_1] \left[a(u)\, b(u) - \frac{a(u^*)}{b(u^*)} \right] v_1$$

Now, let $u \in [u_1, u_+)$. Then according to (1.4) and (1.8)

$$a(u)\, b(u) - \frac{a(u^*)}{b(u^*)} < a(u)\, b(u) - A / B < 0$$

Hence, we have $\psi(u, 0) < 0$ for $u \in [u_1, u_+)$

For any fixed value of u the function $\psi(u, \sigma_1)$ depends linearly on σ_1 and has different signs of the upper and lower boundaries for $u \in [u_1, u_+)$. Hence, a line $\sigma_1 = K(u)$ is found on $[u_1, u_+)$ within E such that $c_1 = K(u)$ we have $\psi < 0$ for $c_1 \in [0, K(u))$ (the domain E_1), and $\psi > 0$ for $c_1 \in (K(u), L(u)]$ (the domain E_2). Evidently

$$\lim_{u \to u_+ - 0} K(u) = 0$$

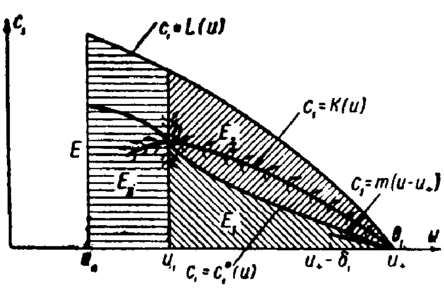

Fig. 1

Because $a' < 0$ for $u \in (u_1, u_+)$ and $\varphi(u, \sigma_1) < 0$, we have $\sigma_1' < 0$ within E_1 from (4.2).

Analogously, we have $\sigma_1' > 0$ within E_2. Let us consider the point $O[u_1, K(u_1)]$. According to (4.2), this point is singular, since $a' = 0$ and $\psi = 0$ there. We establish, by a method mentioned in [5], that the point O is a saddle point, and the slope of the separatrix has two nonzero values of different sign. Let us consider the separatrix $\sigma_1^\circ(u)$ issuing from the point O with a negative slope. Evidently $\sigma_1^\circ(u)$ will fall into the domain E_1 upon motion to the right, and cannot intersect $\sigma_1 = 0$ in (u_1, u_+) because of the uniqueness theorem, nor $\sigma_1 = K(u)$ because $K'(u)$ is finite, and the slope of the integral lines (0.8) is $-\infty$ for $\sigma_1 = K(u) - 0$. Therefore, $\sigma_1^\circ(u)$ drops to the point $O_1(u_+, 0)$. The latter is also singular.

Let us prove that $\sigma_1^\circ(u_+) < 0$. Let us consider the function of two variables

$$Z(u, m) = \frac{a'b\,[\lambda_1 (m + 1)(u - u_+) - v_1]}{(\lambda_2 - \lambda_1 b)\, am + \lambda_2 - \lambda_1 ab + (a - 1)\, bv_1 (u - u_+)^{-1}}$$

Evidently

$$\lim_{\substack{u \to u_+ - 0 \\ m \to 0}} Z(u, m) = 0 \qquad (v_1(u_+) = 0)$$

$$\lim_{u \to u_+ - 0} \left[\lambda_2 - \lambda_1 ab + \frac{(a - 1)\, bv_1}{u - u_+} \right] = \lambda_2 - \lambda_1 a(u_+)\, b(u_+) + [a(u_+) - 1]\, b(u_+)\, v_1'(u_+) =$$

$$= \frac{a(u_+)\,[1 - b^2(u_+)]\,[\lambda_2 - \lambda_1 b(u_+)]}{a - b^2} > 0$$

where $v_1'(u_+)$ from (0.9). Therefore, there exist a $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$Z(u, m) < 1 \text{ for } u_+ - \delta_1 < u < u_+ \text{ and} - \delta_2 < m < 0 \qquad (4.3)$$

Let us chose an arbitrary value $m \in (-\min\{\delta_2, c_1^\circ(u_+ - \delta_1)/\delta_1\}, 0)$. It follows from the condition $m > -c_1^\circ(u_+ - \delta_1)/\delta_1$ that

$$c_1^\circ(u_+ - \delta_1) > -m\delta_1$$

Upon further motion to the right the line $c_1^\circ(u)$ cannot intersect a segment of the line $c_1 = m(u - u_+)$ in $(u_+ - \delta_1, u_+)$. In fact, since $0 > m > -\delta_2$, then (4.3) is satisfied. Multiplying (4.3) by m, we see that the intrinsic slope of the considered segment m is less than the slope of the intergral line at any of its points $mZ(u, m)$. Therefore

$$c_1^{\circ\prime}(u_+) \leqslant m < 0$$

Q.E.D.

Finally, let us continue $c_1^\circ(u)$ from the point O_1 towards the left. Let the part of E not in E_1 and E_2 be denoted by E_3 (Fig.1). Exactly as has been done in studying the domains E_1 and E_2, we see that because of the change in sign of a' when u goes through u_1, we have $c_1' < 0$ for

$$c_1 \in (\max\{0, K(u)\}, L(u))$$

and we have $c_1' > 0$ for $c_1 \in (0, K(u))$ for those u for which $K(u) > 0$. The line $c_1^\circ(u)$ does not intersect $L(u)$ in E_3 since the slope of the integral lines (0.7) is zero on $L(u)$, and $L'(u) < 0$ for $u > u_0$.

Let us prove that $c_1^{\circ\prime}(u) < 0$ for $u \in [u_0, u_1)$. By virtue of continuity, a $\delta > 0$ is found such that we have $L(u) > K(u) > 0$ for $u \in [u_1 - \delta, u_1]$.

Evidently the line $c_1^\circ(u)$ will turn out to be higher than $K(u)$ for $u \in [u_1 - \delta, u_1)$.

There remains to prove that $c_1^{\circ\prime} < 0$ upon further motion to the left.

Let us assume the opposite. This means that at some point $u_2 \in [u_0, u_1 - \delta)$ either $c_1^{\circ\prime} = 0$ or $c_1^{\circ\prime} = \infty$. Let u_2 be the point closest to $u_1 - \delta$ with the mentioned singularity. The case $c_1^{\circ\prime} = 0$ is impossible since we have $c_1^{\circ\prime} < 0$ on $(u_2, u_1 - \delta)$, from which $c_1^\circ(u_2) > c_1^\circ(u_1 - \delta) > 0$, while $c_1^\circ(u_2) < L(u_2)$ and, therefore, we have $a'\varphi[u_2; c_1(u_2)] \neq 0$.

Let us prove that the case $c_1^{\circ\prime} = \infty$ is also impossible. Indeed, if $u_2 > u_0$, then we have $c_1' = -\infty$ for $c_1 = K(u_2) + 0$ and $c_1 = K(u_2) - 0$, where $|K'(u_2)| < \infty$. If $u_2 = u_0$, then $K(u_0) = -\infty$, and $c_1(u_0) > 0$. Therefore, no integral line intersects the line $c_1 = K(u)$ in $[u_0, u_1)$ for right-to-left motion. Thus, the existence of the solution (0.8) $c_1 = c_1^\circ(u)$ satisfying the following conditions:

$$\begin{gathered} c_1^\circ(u_+) = 0, \quad c_1^{\circ\prime}(u) < 0, \quad u \in [u_0, u_+], \quad 0 < c_1^\circ(u) \\ u \in [u_0, u_+), \qquad c_1^\circ(u) < \frac{v_1}{\lambda_1} + u_+ - u, \qquad u \in [u_0, u_+) \end{gathered} \qquad (4.4)$$

is proved.

Substituting $\lambda = \lambda_1$, $c = c_1^\circ(u)$, $c' = c_1^{\circ\prime}(u)$, $v = v_1(u)$ into (0.4), we find $\beta(u)$. It follows from (4.4) that $\beta(u) > 0$ for $u \in [u_0, u_+]$. Moreover, substituting $v = v_1(u)$ and $v' = v_1'(u)$ into (0.4) we obtain $f(u)$. Since

$$v_1' - \lambda_1 < 0, \quad c_1 > 0, \quad v_1 > 0, \quad u \in (u_0, u_+)$$

we obtain $f(u) > 0$ in the mentioned interval. Because of

$$v_1'(u_+) - \lambda_1 < 0, \quad c_1^\circ(u_+) = 0, \quad c_1^{\circ\prime}(u_+) < 0, \quad v_1(u_+) = 0, \quad v_1'(u_+) < 0$$

we obtain $f(u_+) > 0$ from (0.4) by L'Hopital's rule. From $v_1'(u_0) = \lambda_1$, $c_1^\circ(u_0) > 0$, $v_1(u_0) > 0$ we obtain $f(u_0) = 0$. Let us complete determining the function $f(u)$ on $[0, u_0)$ by setting it identically equal to zero, and $\beta(u)$ also in an arbitrary way under the condition of it being positive and continous.

Therefore, a u_+ has been found, and also a $f(u)$ and $\beta(u)$ have been found in $[u_0, u_+]$ satisfying the constraints (0.3), for which the system (0.4), (0.5) has at least two solutions

$$\lambda = \lambda_1, \quad v = v_1(u), \quad c = c_1^\circ(u)$$

$$\lambda = \lambda_2, \quad v = v_2(u) = b(u)\, v_1(u), \quad c = c_2(u) = a(u)\, c_1^\circ(u)$$

BIBLIOGRAPHY

1. Zel'dovich, N.B., K teorii rasprostraneniia plameni (On the theory of flame propagation). Zh.fiz.Khim., Vol.22, №1, 1948.
2. Kanel', N.I., O statsionarnykh resheniiakh dlia sistemy uravnenii teorii goreniia (On stationary solutions for the system of equations of combustion theory). Dokl.Akad.Nauk SSSR, Vol.149, № 2, 1963.
3. Barenblatt, G.I. and Zel'dovich, Ia.B., Ob ustoichivosti rasprostraneniia plameni (On the stability of flame propagation). *PMM*, Vol.21, № 6, 1957.
4. Gel'fand, I.M., Nekotorye zadachi teorii kvazilineinykh uravnenii (Some problems of the theory of quazi-linear equations). Usp.mat.Nauk, Vol.14, № 2, 1959.
5. Stepanov, V.V., Kurs differentsial'nykh uravnenii (A Course on Differential Equations). Gostekhizdat, 1945.

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